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Dynamic time-scale for Lagrangian-averaged subgrid-scale models based on Rice's formula

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Abstract

The dynamic formulation of Smagorinsky's subgrid-scale model for Large Eddy Simulations (LES) requires averaging to avoid instability due to extreme fluctuations. For complex-geometry flows, a Lagrangian time-averaging approach is often useful [see Meneveau, Lund, and Cabot, JFM 319 (1996)]. However, an ad-hoc choice of the relaxation time-scale must be made, often based on resolved strain-rates and stresses at the grid-scale. Recently, Park and Mahesh [Phys. Fluids 21, 065106 (2009)] proposed the attractive notion of using statistics of the Germano-identity error along pathlines to determine a time-scale dynamically. We adopt this concept, but determine the time-scale using Rice's formula to estimate the time between mean-crossings of the error signal. We set the averaging time-scale to be a multiple (4x) of this value. The approach requires accumulating Lagrange-averages of the square error and its time-derivative squared, which is done using the Eulerian approximation as proposed in the original model. LES of channel flow is analyzed and, for validation, LES of flow through an array of wall-mounted cubes is compared with experimental results. It is argued that the resulting model is not entirely dynamic, since the factor relating the averaging time-scale and the Taylor microscale must be prescribed. In agreement with the results of Park and Mahesh, we find the scaling of the dynamic time-scale to be superior to the original time-scale.

Introduction

One of the most successful and popular subgrid-scale models for Large Eddy Simulations (LES) of turbulent flows has been the dynamic Smagorinsky model [3]. When the Smagorinsky model is combined with Germano's identity [3] to represent effects of the unresolved motions, the expression becomes overconstrained and an error-minimization procedure is needed [4] to obtain the model coefficient. While some localized versions of the dynamic model have been used, most practitioners find that the dynamic coefficient must be determined subject to an error minimization that also involves some kind of averaging. As has been reviewed on several occasions [6, 9, 12], for flows that possess directions of statistical homogeneity, spatial averaging is appropriate. For complex-geometry flows with no spatial directions of homogeneity, temporal averaging can be performed, but to comply with Galilean invariance the averaging should take place in a frame moving with the fluid, i.e. Lagrangian averaging [7].

For LES of high Reynolds number boundary layer flows that do not resolve the viscous region near the wall ("wall-modeled LES"), it has been found that scale-dependence should be taken into account [10], leading to the introduction of the Lagrangian-averaged Scale Dependent (LASD) dynamic model [1]. In the LASD model the error is minimized along particle pathlines using a backward-in-time exponential weighting function that is characterized by a Lagrangian time-scale T. In Refs. [1, 7], the time-scale was set in an empirical fashion using dimensional arguments, as well as practical considerations, based on the Lagrangian averages themselves. In a sense, the approach involved a "non-dynamic" element, since one could select the averaging time-scale based on ad-hoc and adjustable parameters. The underlying idea of the original model of Refs. [1, 7] was that the model parameters to be used in the specification of the eddy-viscosity would not contain adjustable parameters, but that the regions over which users choose to enforce the Germano identity would by necessity involve user-selected decisions and thus not be fully dynamic. In the LASD model the time-scale T is used to define this averaging region.

Recently, aiming to address the ad-hoc specification of T in the original Lagrangian dynamic model as well as an observed mismatch with the autocorrelation time-scale of the dynamic error, Park and Mahesh [8] proposed to use the simulated autocorrelation structure of the error in the Germano identity to select the averaging time-scale. This idea has the attractive feature that the selected time-scale would dynamically adjust to the time-scales of the flow. They proposed to determine the correlation time-scale by evaluating the two-time correlation at a small number of time delays, fitting a parabola centered at zero time displacement, and then extrapolating to determine a characteristic time-scale of the process as the time-delay at which the parabola crosses zero correlation. More recently, the approach has been further applied in other flows [14].

In this work, we extend the ideas of Ref. [8] and recall that the extrapolated zero-crossing of the autocorrelation function is related to the temporal Taylor microscale of the process, which can be determined quite naturally by evaluating the variance of the time derivative of a process. As a further motivation for invoking the Taylor microscale, we recall that for Gaussian processes, there is a one-to-one relationship between the average frequency of level-crossings of a signal and the Taylor microscale [11]. This relationship has been used in the context of turbulent flows in Ref. [13].

Formulation

In the dynamic model, stress and strain-rate like tensors L_{ij} and M_{ij} are defined based on the resolved velocity field and strain-rate tensors, at the grid-scale Δ ($\tilde{\ }$) and test-filter scale 2Δ ($\hat{\ }$), as $L_{ij} = \widehat{\tilde{u}_i\tilde{u}_j} - \hat{\tilde{u}}_i\hat{\tilde{u}}_j$ and $M_{ij} = 2\Delta^2\left[\widehat{|\tilde{S}|\tilde{S}_{ij}} - 4|\hat{\tilde{S}}|\hat{\tilde{S}}_{ij}\right]$. The error in the Germano identity is then defined according to $\langle e_{ij}e_{ij}\rangle = \langle\left(L_{ij} - c_s^2 M_{ij}\right)^2\rangle$. The Lagrangian time-averaged versions of appropriate contractions of the various tensors are denoted as $\langle L_{ij}M_{ij}\rangle \equiv \mathcal{I}_{LM} = \frac{1}{T}\int_{-\infty}^{t} L_{ij}M_{ij}(t')\exp\left(-\frac{t-t'}{T}\right)dt'$, and similarly for $\mathcal{I}_{MM}$. In practice, these averages are updated at time-step $n+1$ based on the values at time-step n according to

$$\mathcal{I}_{LM}^{n+1}(\mathbf{x}) = (1-\varepsilon)\mathcal{I}_{LM}^{n}(\mathbf{x} - \tilde{\mathbf{u}}\Delta t) + \varepsilon L_{ij}^{n+1}(\mathbf{x})M_{ij}^{n+1}(\mathbf{x}), \quad (1)$$

where $\varepsilon = \frac{\Delta t}{T}/\left(1 + \frac{\Delta t}{T}\right)$, thus avoiding the need for backward integration in time. The upstream 'off-grid' values can be evaluated using interpolation of various orders; we use trilinear interpolation [7]. The dynamic coefficient is then given by $c_s^2 = \mathcal{F}_{LM}/\mathcal{F}_{MM}$. In the original model [7], the time-scale T was set according to $T = T_{LM,MM} = \theta\Delta(\mathcal{F}_{LM}\mathcal{F}_{MM})^{-1/8}$, and $\theta = 1.5$ was selected in an ad-hoc fashion based on analysis of DNS. This choice of T has the advantage that when $\mathcal{F}_{LM}$ decreases to zero with a tendency to becoming negative (something one wishes to avoid to prevent numerical instabilities), the time-scale increases rapidly, hence preventing $\mathcal{F}_{LM}$ from ever becoming negative.

The analysis of Ref. [8] showed that for channel flow, the time-scale $T_{LM,MM}$ differed significantly from the autocorrelation time-scale of the error signal $\langle e_{ij}e_{ij}\rangle(t)$ in a Lagrangian frame. Defining $E = \langle e_{ij}e_{ij}\rangle$, they consider the autocorrelation function of the process $E(t)$ in (Lagrangian) time, $\rho_E(\tau)$, as a means of calculating an appropriate time-scale T. Near $\tau = 0$, the correlation function of a sufficiently differentiable process can always be approximated with a parabola

$$\rho_E(\tau) \approx 1 - \left(\frac{\tau}{T_{SC}}\right)^2, \tag{2}$$

and the intersect of this parabola with $\rho = 0$ defines a time-scale T_{SC}. This time-scale is termed 'surrogate' by Park & Mahesh since they evaluate the correlation function at every point using practically-motivated approximate techniques that are numerically efficient. Use of this time-scale was shown in both [8] and [14] to lead to very good performance in LES. The selection of cT_{SC} as a time-scale (with $c = 1$) is based on assuming that $c = 1$ is a natural choice. However, recognizing that the autocorrelation of the error reaches zero at significantly longer time-scales than T_{SC}, the choice of $c = 1$ is neither unique nor inevitable. It is therefore important to recognize that an empirically-motivated (non-dynamic) choice has been made in [8] to select this time-scale. However, the experience of [8] and [14] shows that this choice makes sense a-posteriori, since the resulting time-scale shows good scaling properties across various flow regions.

It is useful to point out that T_{SC} is proportional to the temporal Taylor microscale of the process,

$$\tau_E^2 = \frac{\langle E'^2\rangle}{\langle (dE'/dt)^2\rangle}, \tag{3}$$

where $E' = E - \langle E\rangle$. Specifically, $T_{SC} = \sqrt{2}\,\tau_E$ since $\rho_E(\tau) \approx 1 - \frac{1}{2}[\langle (dE'/dt)^2\rangle / \langle E'^2\rangle]\tau^2$. In this work, we recall another interpretation of the Taylor microscale as being related to the average zero-crossing of a signal [11, 13]. In particular, for a Gaussian process, the Rice formula [11] states that the average zero-crossing time-scale T_z of a signal E is given by

$$T_z = \left(\frac{\pi^2\langle E^2\rangle}{\langle (dE/dt)^2\rangle}\right)^{1/2} = \pi\,\tau_E. \tag{4}$$

In this expression, the averaging of E^2 and $(dE/dt)^2$ also must be understood, and be performed, using Lagrangian averaging. An equivalent expression for mean-crossings has an additional prefactor which is approximately unity for signals with a standard deviation much larger than its mean [11]. Since $E(t)$ is non-negative with large standard deviation, we use Eq. 4 to estimate the mean-crossing time-scale.

We have performed various tests using the dynamically determined time-scale to evaluate such averages, and then using these averages to determine the time-scale (numerically, in an explicit approach). Reassuringly, such tests have shown that using averages to compute a time-scale that itself depends on those averages does not cause instability. Furthermore, the derivative in $(dE/dt)^2$ is a material derivative, and to evaluate it, we use the 'upstream' earlier values of average square error.

Our initial experimentations have shown that if an averaging time-scale $T = T_z$ is used, there remains a significant proportion of points on which the numerator $\mathcal{F}_{LM}$ becomes negative and thus perhaps longer periods of averaging may be required. Since typically expected fluctuations of the dynamic error will involve an 'up' and 'down' over a time $T_z = \pi\,\tau_E$, if one wishes to obtain a more or less 'converged' value of the average, one may need several such ups and downs to obtain something close to an average value. Assuming we wish to have (say) two typical 'cycles' (i.e. 4 zero-crossings), one may choose an averaging time-scale $T = T_{EZ} = 4T_z = 4\pi\,\tau_E$. Note, however, that this choice is again 'non-dynamic'. Also, it is significantly larger than the time-scales used by [8] and [14], although they scale in the same way. Finally, we point out that the Rice theorem holds for Gaussian signals, whereas the square error in the Germano-identity, being a square quantity as well as an intermittent small-scale variable in turbulence, presents highly non-Gaussian statistics. Our tests using synthetic signals have shown that Rice's formula remains approximately valid even if the process $E(t)$ is not Gaussian.

Results

In this section, the dynamic time-scale approach using $T = T_{EZ} = 4\pi\,\tau_E$ is applied to LES of high Reynolds number atmospheric boundary layer flow and to flow over an array of cubes. The numerical code has been described in various prior publications [1, 2, 10]. It uses pseudo-spectral discretization in horizontal planes (periodic BC), and 2nd-order finite differencing in the vertical, with a stress-free lid at the top and a standard log-law extrapolation to replace the no-slip BC at the bottom wall. Time-advancement is done using 2nd-order Adams-Bashforth. The Lagrangian averages are updated once every 5 time-steps only, as in prior applications [1]. We use a further variant of the dynamic model, namely the scale-dependent version [1, 10] to account for changes in coefficient with scale as the surface is approached. Test filtering is done in horizontal planes using spectral-cutoff filters at 2Δ and 4Δ. The only difference with the approach followed in [1] is the choice of averaging time-scale, which is evaluated according to Eq. 4 (with a factor 4x) and evaluating averages of E^2 and $(dE/dt)^2$ using the method as in Eq. 1. The flow is forced by means of an applied pressure gradient. The simulation is for flow at very high Reynolds number and so the molecular viscosity is set to zero. The roughness length for the bounding surface is set to $z_0 = 10^{-4}H$ where H is the height of the domain. We compare results obtained using the time-scale T_{EZ} and $T_{LM,MM}$.

The results (not shown) for the boundary layer flow are such that the mean velocity profiles, as well as profiles of resolved Reynolds stresses are almost the same when using the traditional or the new time-scale. There are some minor differences in average subgrid-scale shear stresses, with the T_{EZ} results leading to slightly smaller SGS shear stress. In terms of the dynamically determined variables themselves, there are clear differences. Figure 1 shows (left panels) the resulting mean coefficient values for both time-scales as function of height (in units of Δ). The new time-scale, being shorter over much of the channel (see middle panels), yields also smaller values of the dynamic Smagorinsky coefficient. Comparing the time-scales in the middle panels, we find similar results as those from Ref. [8] and [14]: the dynamic time-scale is more representative of the local turn-over time-scale compared to the behavior exhibited by $T_{LM,MM}$, which near the surface becomes very small due to large shear that makes M_{ij}^2 very high, as pointed out in Ref.

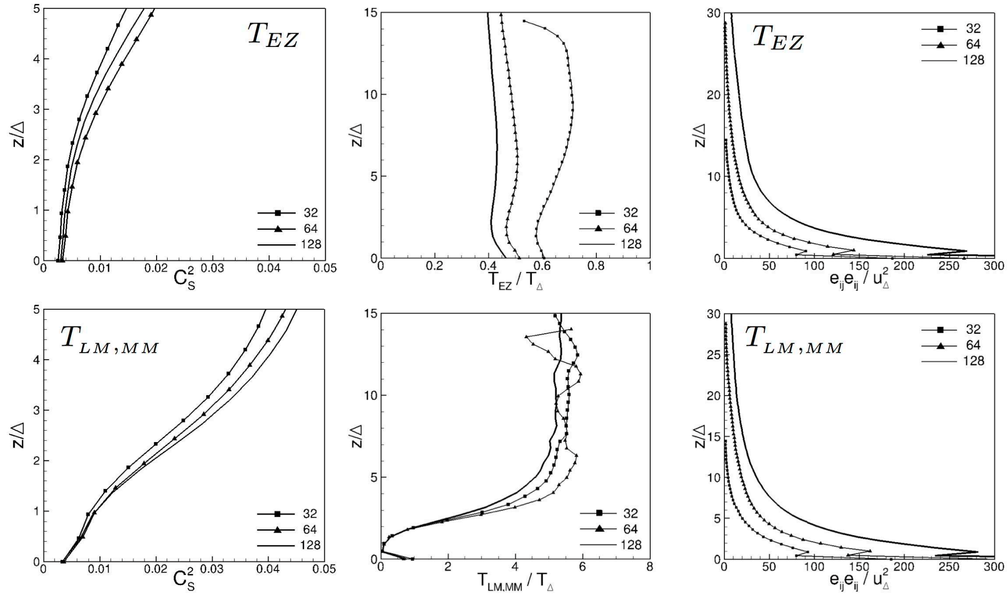

Figure 1: Left panels: profiles of dynamic coefficient. Middle panels: dynamically computed time-scale, scaled by the local reference turn-over time-scale $T_\Delta = \Delta/u_\Delta = (\kappa z)/u_* \times (\Delta/\kappa z)^{2/3}$. Right panels: average Germano-identity square error. Top panels use the dynamic time-scale $T_{EZ} = 4\pi\tau_E$ while bottom panels use the traditional time-scale $T_{LM,MM}$.

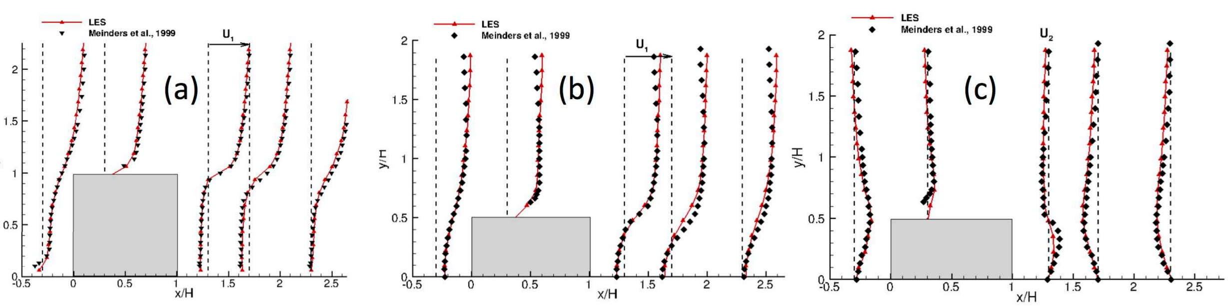

Figure 2: Mean velocity profiles predicted by LES using the dynamic time-scale (triangles) and experimental data from Ref. [5]. (a) Shows side view across center of cubes for streamwise mean velocity, (b) shows top view of half the domain, a cut through half the cube, streamwise and (c) transverse velocity.

[8]. Interestingly, examining the right panels, we also find that the dynamic error is slightly lower for the dynamic time-scale, as compared to the traditional model, although the differences are small.

Next, we consider a flow with a fully complex spatial structure: flow over a periodic array of wall-mounted cubes. For this application, objects in the flow are represented using a variant of the immersed boundary method, as detailed in Ref. [2]. Four cubes are explicitly modeled, and the geometry follows that of Ref. [5] whose data we use to compare to LES predictions. The LES resolves a 2×2 cube array with periodic boundary conditions, and uses a very coarse mesh with $64\times 64\times 29$ grid points (8 points are used per cube edge) in order to provide a stringent test of model and code. The domain size is $8h\times 8h\times 3.5h$, where h is the size of the cubes. Boundary conditions on the cubes are highly approximate in the sense that we use the classic log-law applied normal to the surface, since we do not resolve the viscous sub-layers on any of the surfaces.

Figure 2 shows mean streamwise and cross-stream velocity profiles at various downstream locations. LES predictions follow the experimental data quite well. Further results associated with the model are shown in Figure 3. The upper left panel shows that the dynamically computed coefficient is close to the standard value $c_s^2 \sim 0.01-0.02$, except in the near-wake region where the coefficient is larger. The time-scale (shown on the top center panel) shows only small variations across the domain. The average square error and square time-derivative all display smooth distributions across the domain, as do the averages of $\mathcal{F}_{LM}$ and $\mathcal{F}_{MM}$.

Conclusions

A follow up study of a dynamic time-scale Lagrangian subgrid-scale model for LES [8] has been undertaken. Connections to the temporal Taylor-scale and mean zero-crossing scale of the error signal generated by the Germano identity have been pointed out. Simulations in (half)-channel high Reynolds number flow show very little differences between the mean velocity and Reynolds stress profiles when compared to the original time-scale. However, the dynamically-computed time-scale displays more uniform and reasonable scaling behavior as function of distance to the ground, in agreement with the findings of Refs. [8, 14]. In terms of added computational expense, compared to accumulating the averages for $\mathcal{F}_{LM}$ and $\mathcal{F}_{MM}$ as in the traditional approach, additionally for the present dynamic time-scale the averages of E^2 and $(dE/dt)^2$ must be accumulated.

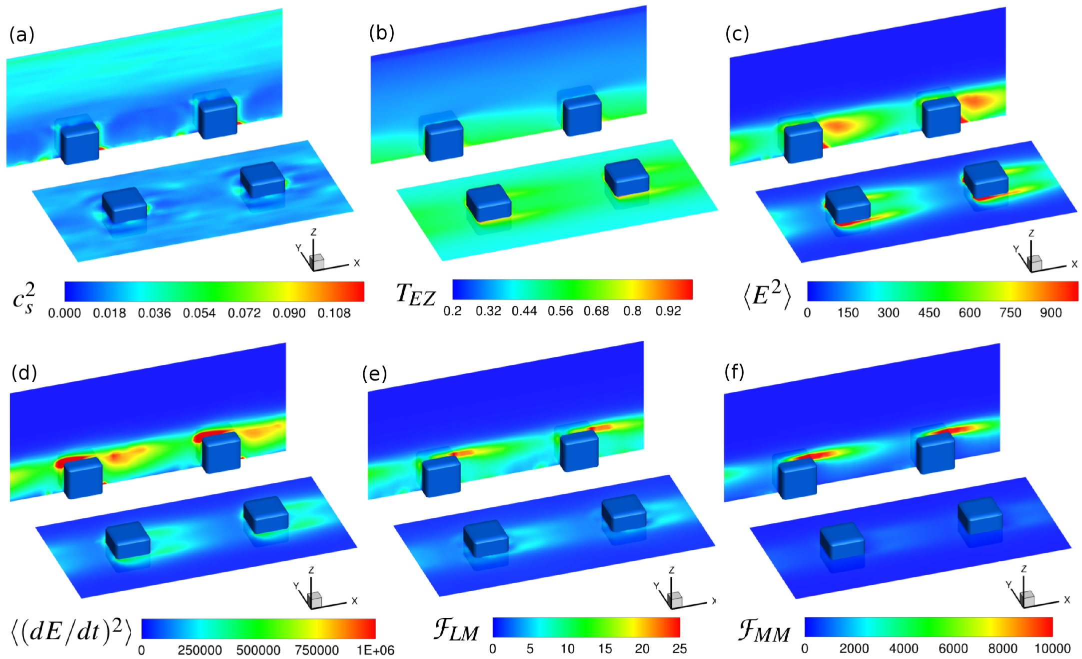

Figure 3: Contours across the domain of (a) the dynamic coefficient, (b) the dynamic time-scale in units of H/u_* where u_* is the friction velocity and H is the domain height, (c) mean square error in units of $(u_*^4)^2$, (d) the squared time derivative of the error in units of $(u_*^5/H)^2$, (e) $\mathcal{F}_{LM}$ in units of u_*^4, and (f) $\mathcal{F}_{MM}$ in units of u_*^4.

Each requires a trilinear interpolation at each point to advance in time (and in addition such interpolation is required to evaluate the Lagrangian time derivative of E). As with the other subgrid values, this is only done every five time steps during the simulation. On average, simulations using the dynamic time-scale took about 13 % longer to run compared to the traditional Lagrangian dynamic model.

Concluding, we remark that the interpretation of the averaging time-scale based on the mean-crossing time for the error signal is conceptually appealing and has the advantage found in Refs. [8, 14] of leading to averaging time-scales that agree better with expected physical eddy-turnover time-scales of the flow. However, the interpretation based on the mean-crossing time also highlights the fact that recourse to a non-dynamic parameter must still be made, both in the present approach (we use $4\pi\tau_E$), as well as in that of Refs. [8, 14] (who selected $\sqrt{2}\,\tau_E$).

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